The spectrum estimation problem

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Contents

The spectrum estimation problem

QUESTION

- ρ is an unknown density operator on $\mathbf{C}^d$
- we have $\rho^{\otimes k}$ at our disposal
- Try to estimate the eigenvalues of ρ
  i.e. estimate $r := \text{spec}(\rho) = (r_1, \ldots, r_d)$
  $\nwarrow \; r_1 \geq \cdots \geq r_d$

The spectrum estimation problem

REALISATION

- Apply a quantum measurement to $\rho^{\otimes k}$
  $\implies$ for example a POVM $(E_i)_i$
- If the outcome is i, give $\hat{r}_i$ $(=output)$ as the estimate of r

GOAL

Find a good measurement, i.e. $\mathbf{P}(|output - r| > \epsilon)$ is small

A brief answer to the question

but a definition first

Yes.

We can find a good measurement to answer the spectrum estimation problem.

To have a quick look at the answer, we need the following definition :

- $\lambda \vdash k$: partiton of k, that is, $\lambda = (\lambda_1, \ldots, \lambda_q)$ for some $q \in \mathbf{N}$, with $\sum \lambda_i = k$ and $\lambda_1 \geq \cdots \geq \lambda_q \geq 1$
- $\lambda \vdash (k, d)$: partition of k into at most d parts, i.e. if $q \leq d$

A brief answer to the question

We can find a POVM $(E_\lambda)_\lambda$ indexed by $\lambda \vdash (k, d)$ that solves the question :

measure $\rho^{\otimes k}$, and if the outcome is λ, give $\bar{\lambda} := (\frac{\lambda_1}{k}, \frac{\lambda_2}{k}, \dots)$ $(=output)$ as the estimate of r

$$\mathbf{P}(\|output - r\|_1 \geq \epsilon) \leq (k+1)^{d(d+1)/2} \exp(-k\epsilon^2/(2\ln 2))$$

Idea of the proof : establish a bound for $\mathrm{tr}(E_\lambda \rho^{\otimes k})$, then use

$$\mathbf{P}(\|output - r\|_1 \geq \epsilon) = \sum_{\substack{\lambda \vdash (k,d) \\ \|\bar{\lambda} - r\|_1 \geq \epsilon}} \mathrm{tr}(E_\lambda \rho^{\otimes k})$$

We are going to construct the POVM, and we first need some representation theory.

- The irreducible representations $\mathcal{U}_\lambda$ of S_k are indexed by $\lambda \vdash k$ integer partitions of k, with

$$\dim \mathcal{U}_\lambda \leq \frac{k!}{\prod_i \lambda_i!}$$

- The irreducible representations $\mathcal{V}_\lambda$ of $SU(d)$ are indexed by $\lambda \vdash (k, d)$ integer partitions of any number k into at most d parts, with

$$\dim \mathcal{V}_\lambda \leq (k+1)^{d(d-1)/2}$$

Representation theory

Schur-Weyl duality

A particular vector space that carries representations of both S_k and $SU(d)$ is $(\mathbf{C}^d)^{\otimes k}$, with the group actions defined as

$$\forall \pi \in S_k, \quad \pi.|i_1 i_2 \dots i_k\rangle = |i_{\pi^{-1}(1)} i_{\pi^{-1}(2)} \dots i_{\pi^{-1}(k)}\rangle$$
$$\forall U \in SU(d), \quad U.|\phi\rangle = U^{\otimes k}|\phi\rangle$$

Theorem (Schur-Weyl duality)

The direct sum decomposition into irreducible representations of $S_k \times SU(d)$, which is multiplicity free :

$$(\mathbf{C}^d)^{\otimes k} = \bigoplus_{\lambda \vdash (k,d)} \mathcal{U}_\lambda \otimes \mathcal{V}_\lambda$$

Schur-Weyl duality

$$(\mathbf{C}^d)^{\otimes k} = \bigoplus_{\lambda \vdash (k,d)} \mathcal{U}_\lambda \otimes \mathcal{V}_\lambda$$

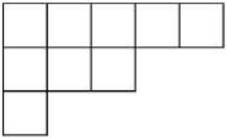
Denote the projection onto $\mathcal{U}_\lambda \otimes \mathcal{V}_\lambda$ by P_λ.

We will see that the POVM $(P_\lambda)_\lambda$ works, i.e. E_λ in the previous slide is defined as P_λ.

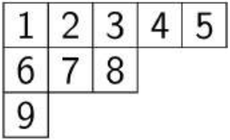
A closer look at $\mathcal{V}_\lambda$

Young tableaux

Given $\lambda \vdash k$ an integer partition of k. ($\lambda = (5, 3, 1)$ in the following examples.)



Young frame of λ



Canonical Young tableau T_λ of λ

A closer look at $\mathcal{V}_\lambda$

Young symmetry operator

$R(T_\lambda) := \{\pi \in S_k : \text{ rows of } T_\lambda \text{ are invariant under } \pi\}$
$C(T_\lambda) := \{\pi \in S_k : \text{ columns of } T_\lambda \text{ are invariant under } \pi\}$

Definition (Young symmetry operator)

$$e(T_\lambda) := \Big(\sum_{\pi \in C_\lambda} \operatorname{sgn}(\pi)\pi\Big)\Big(\sum_{\pi \in R_\lambda} \pi\Big)$$

$e(T_\lambda)^2 = re(T_\lambda)$ for some integer r, so

$$p(T_\lambda) := \frac{e(T_\lambda)}{r}$$

verifies $p(T_\lambda)^2 = p(T_\lambda)$.

A closer look at $\mathcal{V}_\lambda$

Now we define $\mathcal{V}_\lambda$...

Definition

For any $\lambda \vdash k$,

$$\mathcal{V}_\lambda := p(T_\lambda).(\mathbf{C}^d)^{\otimes k}$$

If we fix an orthonormal basis $(|1\rangle, |2\rangle, \ldots, |d\rangle)$ of $\mathbf{C}^d$:

- $|\mathbf{i}\rangle := |i_1 \ldots i_k\rangle$ for any $\mathbf{i} = (i_1, \ldots, i_k) \in [d]^k$
- $(|\mathbf{i}\rangle)_\mathbf{i}$ forms an orthonormal basis of $(\mathbf{C}^d)^{\otimes k}$
- so $\mathcal{V}_\lambda = \mathrm{Span}\left(p(T_\lambda).|\mathbf{i}\rangle\right)$
- $w(\mathbf{i}) := (w_1(\mathbf{i}), \ldots, w_d(\mathbf{i}))$ where $w_l(\mathbf{i}) = \#\{s \in [k] : i_s = l\}$

A closer look at $\mathcal{V}_\lambda$

$\mathcal{V}_\lambda = 0$ if $\lambda \nvdash (k, d)$

Given $\lambda \vdash k$. We say $w(\mathbf{i})$ is majorised by λ ($w(\mathbf{i}) \prec \lambda$) if

$$\forall l \in [d-1],\ \sum_{i=1}^{l} w_i(\mathbf{i}) \leq \sum_{i=1}^{l} \lambda_i; \quad \sum_{i=1}^{d} w_i(\mathbf{i}) = \sum_{i=1}^{d} \lambda_i$$

Observation

If $w(\mathbf{i}) \nprec \lambda$, then $p(T_\lambda).|\mathbf{i}\rangle = 0$.

Therefore, if $\lambda \nvdash (k, d)$, then $\mathcal{V}_\lambda = \text{Span}(p(T_\lambda).|\mathbf{i}\rangle) = 0$

"Therefore" : since if $\lambda \nvdash (k, d)$, we have $w(\mathbf{i}) \nprec \lambda$ for any $\mathbf{i}$

So it works

Write $\rho^{\otimes k}$: we will bound $\mathrm{tr}(P_\lambda \rho^{\otimes k})$ later

Let ρ be a density operator on $\mathbf{C}^d$.

- $r = \mathrm{spec}(\rho) = (r_1, \dots, r_d)$
- $(|1\rangle, \dots, |d\rangle)$ corresponding orthonormal eigenbasis of $\mathbf{C}^d$

Therefore, $\rho^{\otimes k} = \sum_{\mathbf{i}} r_{\mathbf{i}} |\mathbf{i}\rangle\langle\mathbf{i}|$, where

$$r_{\mathbf{i}} := r_{i_1} \cdots r_{i_k} = r_1^{w_1(\mathbf{i})} \dots r_k^{w_k(\mathbf{i})}$$

Given $\lambda \vdash (k, d)$. $P_\lambda |\mathbf{i}\rangle = 0$ if $w(\mathbf{i}) \not\prec \lambda$, so

$$P_\lambda \rho^{\otimes k} = \sum_{\mathbf{i}} r_{\mathbf{i}} P_\lambda |\mathbf{i}\rangle\langle \mathbf{i}| = \sum_{\mathbf{i}\colon w(\mathbf{i}) \prec \lambda} r_{\mathbf{i}} P_\lambda |\mathbf{i}\rangle\langle \mathbf{i}|$$

$$\leq r_1^{\lambda_1} \cdots r_d^{\lambda_d} \sum_{\mathbf{i}\colon w(\mathbf{i}) \prec \lambda} P_\lambda |\mathbf{i}\rangle\langle \mathbf{i}|$$

Therefore,

$$\begin{aligned} \mathrm{tr}(P_\lambda \rho^{\otimes k}) &\leq r_1^{\lambda_1} \cdots r_d^{\lambda_d} \sum_{\mathbf{i}\colon w(\mathbf{i}) \prec \lambda} \mathrm{tr}(P_\lambda |\mathbf{i}\rangle\langle \mathbf{i}|) \\ &\leq r_1^{\lambda_1} \cdots r_d^{\lambda_d}\, \mathrm{tr}(P_\lambda) \\ &= r_1^{\lambda_1} \cdots r_d^{\lambda_d}\, \dim(\mathcal{U}_\lambda) \dim(\mathcal{V}_\lambda) \end{aligned}$$

So it works

Bound $\mathrm{tr}(P_\lambda \rho^{\otimes k})$

By using the bounds on $\dim(\mathcal{U}_\lambda)$ and $\dim(\mathcal{V}_\lambda)$, we get

$$\mathrm{tr}(P_\lambda \rho^{\otimes k}) \leq (k+1)^{d(d-1)/2} \exp\left(\frac{-k\|\bar{\lambda} - r\|_1^2}{2\ln 2}\right)$$

$\uparrow$

This bounds the probability to have the outcome λ (so the output $\bar{\lambda}$) after measuring $\rho^{\otimes k}$.

So it works

we prove the result shown in the beginning

We now have

$$\mathbf{P}(\|output - r\|_1 \geq \epsilon) = \sum_{\substack{\lambda \vdash (k,d) \\ \|\bar{\lambda} - r\|_1 \geq \epsilon}} \operatorname{tr}(P_\lambda \rho^{\otimes k})$$

$$\leq (k+1)^{d(d-1)/2} \exp\left(\frac{-k\epsilon^2}{2\ln 2}\right) \left|\{\lambda \vdash (k,d) : \|\bar{\lambda} - r\|_1 \geq \epsilon\}\right|$$

By using

$$\left|\{\lambda \vdash (k,d) : \|\bar{\lambda} - r\|_1 \geq \epsilon\}\right| \leq \left|\{\lambda \vdash (k,d)\}\right| \leq (k+1)^{d-1}$$

we recover the result shown in the beginning of this seminar :

$$\mathbf{P}(\|output - r\|_1 \geq \epsilon) \leq (k+1)^{d(d+1)/2} \exp(-k\epsilon^2/(2\ln 2))$$

The result in a more succinct way

It means for any ϵ', there is an $k_0 > 0$ such that for all $k \geq k_0$,

$$\mathbf{P}(\|output - r\|_1 \geq \epsilon) < \epsilon',$$

or equivalently,

$$\sum_{\substack{\lambda \vdash (k,d) \\ \|\bar{\lambda} - r\|_1 < \epsilon}} \mathrm{tr}(P_\lambda \rho^{\otimes k}) \geq 1 - \epsilon'.$$

Reference

The Spectra of Quantum States and the Kronecker Coefficients of the Symmetric Group

- Matthias Christandl1, Graeme Mitchison, 2005